V_∞ = bubble terminal velocity, cm./sec.
ρ = density, g./cc.
σ = surface tension, dynes/cm.

Subscripts

l = liquid
v = vapor

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A Mathematical Analysis of the Surface Temperature Variation in Heat Transfer Experiments

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McAdams (*1*) has presented a summary of theoretical and experimental information concerning the transfer of heat between a circular tube and a fluid flowing normal to the axis of the tube. The experimental data are correlated as average heat transfer coefficients defined in terms of the total heat flux per unit area and the average temperature difference between the surface and the bulk fluid. However, for tubes of large wall thickness or low conductivity, the surface temperature variation can be significant with respect to the maximum temperature difference between the surface and the bulk fluid, and thus the average temperature difference can be experimentally uncertain.

In general the calculation of the temperature variation along a tube surface, when the thermal resistance of the tube wall is comparable to the thermal resistance of the fluid, requires the solution of the heat conduction equation in both phases. As a first approximation to the solution, the thermal resistance of either phase may be assumed to be zero with the result that the surface temperature is found to be constant. As a second approximation, the thermal resistance of the fluid may be assumed to be nearly zero and may therefore be represented by a position-dependent surface heat transfer coefficient. The problem then is reduced to one of heat conduction in the tube wall for an arbitrary heat transfer coefficient. The solution to this problem is given for a tube with a heat generating core.

Carslaw and Jaeger (*3*) have given the solution to some steady heat conduction problems with position-dependent boundary conditions; however, an alternate series solution may be found if the position-dependent coefficient is expanded in the angular parts of the terms constituting the general solution of Laplace's equation.

THE FORMAL SOLUTION

The problem is to find the steady temperature distribution in an infinite hollow cylinder of radius R_2, containing a heat generating core of radius R_1, for any preassigned local heat transfer coefficient (Figure 1). The coefficient $h(\theta)$ is permitted to vary with circumference along the surface $R = R_2$ and in general is given by

$$h(\theta) = \sum_{p=0}^{q} a_p \cos p\theta \qquad (1)$$

where q is any positive integer and where the flow direction is $\theta = 0$.

Laplace's equation for steady conduction may be written in cylindrical coordinates as

$$\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \qquad (2)$$

when

$$\frac{\partial T}{\partial Z} = 0$$

The boundary conditions express the symmetry of the problem and the requirement that the flux be continuous at the surfaces $R = R_1$ and $R = R_2$. These conditions are given by

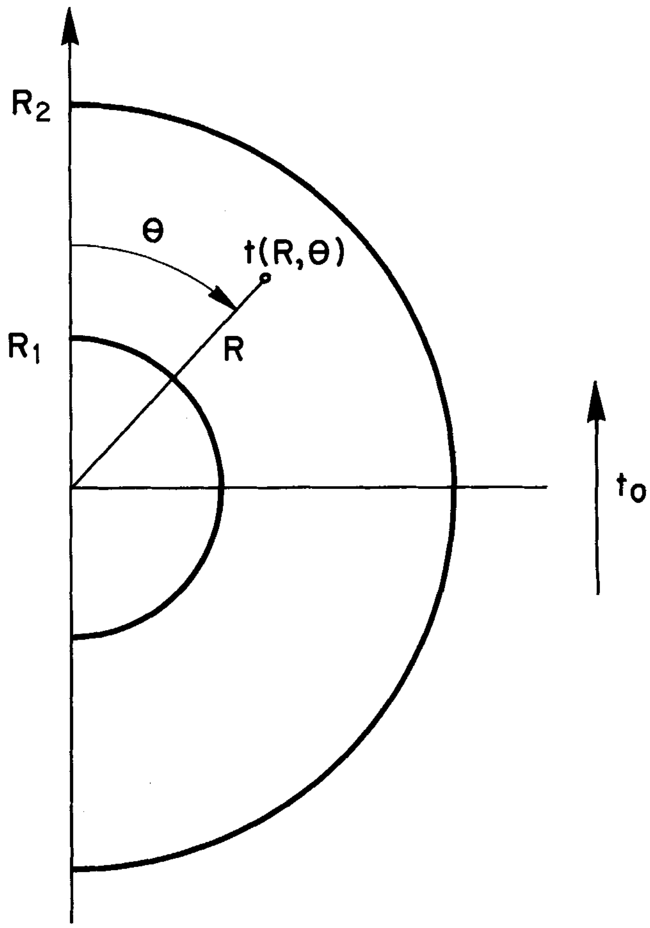

Fig. 1. Cross section of infinitely long hollow cylinder.

$$\frac{\partial T}{\partial \theta}(r, 0) = \frac{\partial T}{\partial \theta}(r, \pi) = 0 \qquad (3)$$

$$\frac{\partial T}{\partial r}(r_1, \theta) = -\frac{R_2 Q_1}{k} \qquad (4)$$

and

$$\frac{\partial T}{\partial r}(1, \theta) = -\frac{R_2 h(\theta)\, T(1, \theta)}{k} \qquad (5)$$

A convenient form of solution is given by the method of separation of variables, as shown by Margenau and Murphy (2). The general solution of Equations (2) and (3) can be written as a superposition of constituent solutions in the following form:

$$T(r, \theta) = C_0 \ln r + D_0 + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n}) \cos n\theta \qquad (6)$$

where

$$C_n, D_n, n = 1, \ldots$$

are arbitrary constants dependent on the additional information contained in the boundary conditions, Equations (4) and (5). The mutual orthogonality of the functions

$$1, \cos\theta, \cos 2\theta, \ldots$$

on the interval $0, \pi$ simplifies the development of a set of recurrence relations among the constants in Equation (6). The procedure is to substitute the series representation of T into Equations (4) and (5), multiply the resulting equations through by $\cos(m\theta)\, d\theta$, and then integrate from 0 to π. In this way Equations (4) and (6) yield

$$m = 0, \quad C_0 = -\frac{R_1 Q_1}{k} \qquad (7)$$

$$m \geqq 1, \quad D_m = r_1^{2m} C_m \qquad (8)$$

while Equations (5) and (6), together with Equation (1) and the relationship

$$2 \cos p\theta \cos n\theta = \cos (n + p)\,\theta + \cos (n - p)\,\theta$$

yield

$$m = 0, \; C_0 = -\frac{R_2}{k} D_0 a_0 - \frac{1}{2}\frac{R_2}{k}\sum_{p=1}^{q} a_p (C_p + D_p) \qquad (9)$$

$$1 \leqq m \leqq q - 1, \; m(C_m - D_m) = -\frac{R_2}{k} D_0 a_m - \frac{1}{2}\frac{R_2}{k}\sum_{p=0}^{m-1} a_p (C_{m-p} + D_{m-p}) - \frac{1}{2}\frac{R_2}{k}\sum_{p=0}^{q} a_p (C_{m+p} + D_{m+p}) - \frac{1}{2}\frac{R_2}{k}\sum_{p=m+1}^{q} a_p (C_{p-m} + D_{p-m}) \qquad (10)$$

$$m = q, \; q(C_q - D_q) = -\frac{R_2}{k} D_0 a_q - \frac{1}{2}\frac{R_2}{k}\sum_{p=0}^{q-1} a_p (C_{q-p} + D_{q-p}) - \frac{1}{2}\frac{R_2}{k}\sum_{p=0}^{b} a_p (C_{q+p} + D_{q+p}) \qquad (11)$$

and $m \geqq q + 1$

$$m(C_m - D_m) = -\frac{1}{2}\frac{R_2}{k}\sum_{p=0}^{q} a_p (C_{m-p} + D_{m-p}) - \frac{1}{2}\frac{R_2}{k}\sum_{p=0}^{q} a_p (C_{m+p} + D_{m+p}) \qquad (12)$$

When $q = \infty$, Equations (9) through (12) reduce to the following equations:

$$m = 0, \quad C_0 = -\frac{R_2}{k} D_0 a_0 - \frac{1}{2}\frac{R_2}{k}\sum_{p=1}^{\infty} a_p (C_p + D_p) \qquad (9a)$$

and

$$m \geqq 1, \quad m(C_m - D_m) = -\frac{R_2}{k} D_0 a_m - \frac{1}{2}\frac{R_2}{k}\sum_{p=0}^{m-1} a_p (C_{m-p} + D_{m-p}) - \frac{1}{2}\frac{R_2}{k}\sum_{p=0}^{\infty} a_p (C_{m+p} + D_{m+p}) - \frac{1}{2}\frac{R_2}{k}\sum_{p=m+1}^{\infty} a_p (C_{p-m} + D_{p-m}) \qquad (10a)$$

The method of solution to be developed includes the case $q = \infty$ and thus it applies to any coefficient $h(\theta)$ that can be expanded in a Fourier cosine series.

If Equations (7) and (8) are used to eliminate C_0 and $C_1, C_2, \cdots$, Equations (9) through (12) [or (9*a*) and (10*a*)] form a nonhomogeneous set of linear equations of infinite order in the unknown expansion coefficients $D_0, D_1, D_2, \cdots$. In contrast to classical problems in heat conduction in which the orthogonality conditions lead to a diagonal set of linear equations in the expansion coefficients, here the equations are not diagonal and the coefficients cannot be found sequentially but must be found simultaneously. These equations may be solved approximately by observing that a necessary condition for the series

$$C_0 \ln r + D_0 + \sum_{n=1}^{\infty} [C_n r^n + D_n r^{-n}] \cos n\theta$$

to converge at all points r, θ in the region $r_1 \leqq r \leqq 1$, $0 \leqq \theta \leqq \pi$ is that

$$\lim_{n \to \infty} (C_n r^n + D_n r^{-n}) \cos n\theta = 0$$

or

$$\lim_{n \to \infty} \left[\left(\frac{r^n}{r_1^n} r_1^{-n} + r^{-n} \right) \cos n\theta \right] D_n = 0$$

or

$$\lim_{n \to \infty} D_n = 0 \qquad (13)$$

Equation (13) can be used in the following way to find improvable approximations to the coefficients $D_0, D_1, D_2, \cdots$: for some positive integer s the coefficients $D_s, D_{s+1}, \cdots$ are set equal to zero and Equations (9) through (12) are then solved for $D_0, D_1, D_2, \cdots, D_{s-1}$. This procedure may be continued for increasing values of s until as many coefficients become constant as are required for a given accuracy.

When $h(\theta)$ is a constant independent of θ, $a_0 = h$, $a_1 = a_2 = \cdots = a_q = 0$ and Equations 7 through 12 yield

$$C_0 = -\frac{R_1 Q_1}{k}, C_1 = C_2 = \cdots = 0$$

$$D_0 = \frac{R_1 Q_1}{R_2 h}, D_1 = D_2 \cdots = 0$$

As a less trivial example of the use of the above method to find the expansion coefficients, suppose that

$$h(\theta) = a + b \cos^2\theta.$$

This particular $h(\theta)$ introduces symmetry into the problem about $\theta = \frac{\pi}{2}$, that is

$$\frac{\partial T}{\partial \theta}\left(r, \frac{\pi}{2} \right) = 0$$

and as a consequence the summation index n in Equation (6) must be restricted to the even integers. Expanding $h(\theta)$ in a Fourier series, we find that

$$a_0 = a + \frac{b}{2}$$

$$a_1 = 0$$

$$a_2 = \frac{b}{2}$$

$$q = 2$$

and consequently Equations (9), (11), and (12) become

$$C_0 = -\frac{R_2}{k} D_0 \left(a + \frac{b}{2} \right) - \frac{R_2}{k}\frac{b}{4}(C_2 + D_2) \tag{14}$$

$$2(C_2 - D_2) = -\frac{R_2}{k} D_0 \frac{b}{2} - \frac{R_2}{k}\left(a + \frac{b}{2} \right) \times (C_2 + D_2) - \frac{R_2}{k}\frac{b}{4}(C_4 + D_4) \tag{15}$$

and

$$m(C_m - D_m) = -\frac{R_2}{k}\left(a + \frac{b}{2} \right) \times (C_m + D_m) - \frac{R_2}{k}\frac{b}{4}(C_{m-2} + D_{m-2}) - \frac{R_2}{k}\frac{b}{4}(C_{m+2} + D_{m+2})$$

$$m = 4, 6, \ldots. \tag{16}$$

For a specific value of s Equations (7), (8), (14), (15), and (16) form a system of s linear equations in the s unknown coefficients $C_0, C_2, \cdots, C_{s-2}, D_0, D_2, \cdots, D_{s-2}$ where $C_s = C_{s+2} = \cdots = D_s = D_{s+2} = \cdots = 0$.

In the case of water flowing normal to a copper tube [R_1 = 1/24 ft., R_2 = 1/12 ft., $2\pi R_1 Q_1$ = 4,000 B.t.u./(hr.)(ft.)], if the heat transfer coefficient is taken to be given by $a = 0$, $h_m = b/2$ = 1,000 B.t.u./(hr.)(sq.ft.)(°F.), the method for computing the expansion coefficients converges rapidly and the solution is given by $T(r, \theta) =$

$$\begin{aligned} &-3.1831 \ln r + 8.4502 \\ &- (1.5261\, r^2 + 9.5379 \times 10^{-2}\, r^{-2}) \cos 2\theta \\ &+ (7.6845 \times 10^{-2}\, r^4 + 3.0017 \times 10^{-4}\, r^{-4}) \cos 4\theta \\ &- (2.5052 \times 10^{-3}\, r^6 + 6.1163 \times 10^{-7}\, r^{-6}) \cos 6\theta \\ &+ \cdots. \end{aligned} \tag{17}$$

Equation (17) predicts a surface temperature variation of approximately 20% of the average difference in temperature between the surface and the bulk of the fluid.

In comparing the calculations for $s = 6$ and $s = 8$, the coefficients C_0, C_2, D_0, D_2 are found to agree to five significant decimal places, while the coefficients C_4, D_4 are found to agree to three significant decimal places. The coefficients in Equation (17) are those for $s = 8$. Furthermore

$$2\int_0^\pi h(\theta)\, T(1, \theta)\, R_2\, d\theta = 2\pi R_2 h_m \left(D_0 + \frac{C_2 + D_2}{2} \right) = 4.00003 \times 10^3 \text{ B.t.u./(hr.)(ft.)},$$ which

closely approximates $2\pi R_1 Q_1 = 4.0 \times 10^3$ B.t.u./(hr.)(ft.). The above result follows from the orthogonality of the functions 1, $\cos\theta$, $\cos 2\theta$, - - - on the interval 0, π.

Equations (7) through (12) also yield the correct solution in the limit as $1/k \to 0$. Equations (10), (11), and (12) show that for $1/k = 0$

$$D_m = C_m \quad m \geqq 1$$

and therefore in view of Equation (8)

$$D_m = C_m = 0 \quad m \geqq 1$$

since $r_1 \neq 1$. As a result, from Equations (7) and (9)

$$C_0 = 0$$

$$D_0 = \frac{R_1 Q_1}{R_2 a_0}$$

where

$$a_0 = h_m = \frac{1}{\pi}\int_0^\pi h(\theta)\, d\theta$$

The solution therefore is

$$T(r, \theta) = \frac{R_1 Q_1}{R_2 a_0}$$

and this solution satisfies the condition that

$$2\pi R_1 Q_1 = 2\int_0^\pi h(\theta)\, T(1, \theta)\, R_2\, d\theta$$

NOTATION

$a_0, a_1, \ldots$ = constants in Equation (1), B.t.u./(hr.)(sq.ft.)(°F.)
a, b = constants, B.t.u./(hr.)(sq.ft.)(°F.)
$C_0, C_1, \ldots$ = constants in Equation (6), °F.
$D_0, D_1, \ldots$ = constants in Equation (6), °F.
h = local heat transfer coefficient, B.t.u./(hr.)(sq.ft.)(°F.)
h_m = average heat transfer coefficient, B.t.u./(hr.)(sq.ft.)(°F.)
k = thermal conductivity, B.t.u./(hr.)(ft.)(°F.)
n, m, p, q, s = integers
Q = heat flux, B.t.u./(hr.)(sq.ft.)
R = radial coordinate, ft.
r = R/R_2
R_1 = core radius, ft.
R_2 = cylinder radius, ft.
t = point temperature in the cylinder, °F.
t_0 = bulk temperature in the fluid, °F.
T = $t - t_0$
Z = axial coordinate, ft.
θ = angular coordinate, rad.

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